Structure of Uncertainty

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countably infinite number of states of the world
example: rational numbers in $[0, 1]$

set of states:

\[ S = \{s_1, s_2, s_3, \ldots\} \]

nature chooses state $s \in S$ at beginning of time

individuals learn slowly more about true state
partition $Q = \{q_1, \cdots, q_N\}$ of $S = \{s_1, s_2, \cdots\}$ is a collection of subsets of $S$ such that

$$ q_i \cap q_j = \emptyset \text{ for } j \neq i \quad \text{and} \quad \bigcup_{j=1}^{N} q_j = S $$

assumption: partitions contain a finite number of elements

example:

$$ S = \{ \text{rationals} \in [0, 0.2), \quad \text{rationals} \in [0.2, 0.6), \quad \text{rationals} \in [0.6, 1] \} $$

$q_1$ $q_2$ $q_3$
example (partition from previous slide):

<table>
<thead>
<tr>
<th>if nature chooses $s =$</th>
<th>then at time $t$ we know that $s \in$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>[0, 0.2)</td>
</tr>
<tr>
<td>0.1</td>
<td>[0, 0.2)</td>
</tr>
<tr>
<td>0.5</td>
<td>[0.2, 0.6)</td>
</tr>
<tr>
<td>0.8</td>
<td>[0.6, 1]</td>
</tr>
<tr>
<td>1</td>
<td>[0.6, 1]</td>
</tr>
</tbody>
</table>
example: time \( t + 1 \) information

\[
S = \{ \begin{array}{l}
q_1 \quad [0, 0.2), \\
q_2 \quad [0.2, 0.6], \\
q_3 \quad (0.6, 0.9), \\
q_4 \quad [0.9, 1]\end{array} \}
\]

then information evolves as follows:

<table>
<thead>
<tr>
<th>if nature chooses ( s = )</th>
<th>time ( t ) info: ( s \in )</th>
<th>time ( t + 1 ) info: ( s \in )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/20</td>
<td>[0, 0.2)</td>
<td>[0, 0.2)</td>
</tr>
<tr>
<td>1/10</td>
<td>[0, 0.2)</td>
<td>[0, 0.2) ]</td>
</tr>
<tr>
<td>1/2</td>
<td>[0.2, 0.6]</td>
<td>[0.2, 0.6]</td>
</tr>
<tr>
<td>3/4</td>
<td>(0.6, 1]</td>
<td>(0.6, 0.9)</td>
</tr>
<tr>
<td>1</td>
<td>(0.6, 1]</td>
<td>[0.9, 1]</td>
</tr>
</tbody>
</table>
information structure at time $t$: $Q_t = \{q_{t1}, \ldots, q_{tN_t}\}$

no forgetting $\Rightarrow$ partition becomes finer over time

sequence of finer partitions $\{Q_0, Q_1, Q_2 \ldots\}$: filtration

each state $s \in S$ defines a unique path through the partition

elements of this filtration:

$$S \supseteq q_0 \supseteq q_1 \supseteq q_2 \cdots$$
Example of a Filtration

\[ q_0 \rightarrow q_1 \rightarrow q_2 \rightarrow \ldots \rightarrow q_4 \]

- At each time step, the state transitions from one event to the next.
- The diagram illustrates the progression of events over time.
**random variable**: function that assigns a real number to each state \( s \in S \)

Random variable \( Y \) is **measurable** with respect to partition \( Q \) if

\[
s_i \in q \text{ and } s_j \in q \implies Y(s_i) = Y(s_j) \quad \text{for all } q \in Q. \tag{1}
\]

**stochastic process**: sequence of random variables

\[\{ Y_0, Y_1, Y_2, \ldots \}\]

stochastic process \( \{ Y_0, Y_1, Y_2, \ldots \} \) is **adapted** to a filtration \( \{ Q_0, Q_1, Q_2 \ldots \} \) if each \( Y_t \) is measurable with respect to \( Q_t \)
<table>
<thead>
<tr>
<th>state $s$</th>
<th>$Q_t$</th>
<th>$X_t$</th>
<th>$Y_t$</th>
<th>$Z_t$</th>
<th>$Q_{t+1}$</th>
<th>$X_{t+1}$</th>
<th>$Y_{t+1}$</th>
<th>$Z_{t+1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/10</td>
<td>[0, 0.2)</td>
<td>4</td>
<td>3</td>
<td>1</td>
<td>[0, 0.2)</td>
<td>3</td>
<td>7</td>
<td>1</td>
</tr>
<tr>
<td>1/20</td>
<td>[0, 0.2)</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>[0, 0.2)</td>
<td>3</td>
<td>7</td>
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<td>[0.2, 0.6)</td>
<td>4</td>
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<td>1</td>
<td>[0.2, 0.6)</td>
<td>4</td>
<td>8</td>
<td>2</td>
</tr>
<tr>
<td>3/4</td>
<td>[0.6, 1]</td>
<td>2</td>
<td>5</td>
<td>1</td>
<td>[0.6, 0.9]</td>
<td>4</td>
<td>9</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>[0.6, 1]</td>
<td>2</td>
<td>5</td>
<td>1</td>
<td>(0.9, 1]</td>
<td>1</td>
<td>10</td>
<td>2</td>
</tr>
</tbody>
</table>
stochastic process \( \{ Y_t, Y_{t+1} \} \) generates the filtration \( \{ Q_t, Q_{t+1} \} \) if:

\[ s_i, s_j \in q_t \iff Y_t(s_i) = Y_t(s_j) \]
A probability measure “prob” is a function from the power set of $S$ to the real numbers with the following properties:

1. $A \subseteq S \implies \text{prob}[A] \geq 0$
2. $\text{prob}[S] = 1$
3. $A_i \cap A_j = \emptyset$ for $j \neq i \implies \text{prob}\left[\bigcup_{n=1}^{\infty} A_n\right] = \sum_{n=1}^{\infty} \text{prob}[A_n]$
Conditional Probabilities

- probability that nature chooses a state $s$ in $q_t$: $\text{prob}(q_t)$
- assume: $\text{prob}(q_t) > 0$
- probability of $q_{t+\tau} \subseteq q_t$ conditional on observing $q_t$:

$$
\text{prob}(q_{t+\tau}|q_t) = \frac{\text{prob}(q_{t+\tau})}{\text{prob}(q_t)}
$$

- we can also write this probability as

$$
\text{prob}(q_{t+\tau}|q_t) = \frac{\text{prob}(q_{t+1})}{\text{prob}(q_t)} \times \frac{\text{prob}(q_{t+2})}{\text{prob}(q_{t+1})} \times \cdots \times \frac{\text{prob}(q_{t+\tau})}{\text{prob}(q_{t+\tau-1})}
$$

$$
\text{prob}(q_{t+1}|q_t) \quad \text{prob}(q_{t+2}|q_{t+1}) \quad \text{prob}(q_{t+\tau}|q_{t+\tau-1})
$$
unconditional expectation of a random variables $Y_t$:

$$E[Y_t] = \sum_{q_t \subseteq Q_t} \text{prob}(q_t) \times Y_t(q_t)$$

conditional:

$$E[Y_{t+\tau}|q_t] = \sum_{q_{t+\tau} \subseteq q_t} \text{prob}(q_{t+\tau}|q_t) \times Y(q_{t+\tau})$$

$E[Y_{t+\tau}|Q_t]$: random variable with realizations

$$\left\{ E[Y_{t+\tau}|q_{1t}], E[Y_{t+\tau}|q_{2t}], \ldots, E[Y_{t+\tau}|q_{N_t}] \right\}$$

usually abbreviate:

$$E_t[Y_{t+\tau}] = E[Y_{t+\tau}|Q_t]$$
consider the following sequence of expectations:

\[
\begin{array}{cccccc}
  t & \cdots & t + \tau_1 & \cdots & t + \tau_2 \\
\end{array}
\]

\[
E\left[E\left[Y_{t+\tau_2} \mid Q_{t+\tau_1}\right] \mid q_t\right] \quad E\left[Y_{t+\tau_2} \mid Q_{t+\tau_1}\right] \quad Y_{t+\tau_2}
\]

\[ E \left[ E[Y_{t+\tau_2} \mid Q_{t+\tau_1}] \mid q_t \right] = \sum_{q_{t+\tau_1} \subseteq q_t} \sum_{q_{t+\tau_2} \subseteq q_{t+\tau_1}} \text{prob}(q_{t+\tau_1} \mid q_t) \text{prob}(q_{t+\tau_2} \mid q_{t+\tau_1}) \sum_{q_{t+\tau_2} \subseteq q_t} \text{prob}(q_{t+\tau_2} \mid q_t) \text{prob}(q_{t+\tau_1}) Y(q_{t+\tau_2}) \\
= \sum_{q_{t+\tau_2} \subseteq q_t} \sum_{q_{t+\tau_1} \subseteq q_t} \text{prob}(q_{t+\tau_1} \mid q_t) \text{prob}(q_{t+\tau_2} \mid q_{t+\tau_1}) Y(q_{t+\tau_2}) \]